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ON THE NUMBER OF EDGE DISJOINT CLIQUES IN GRAPHS OF GIVEN SIZE

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In this paper, we prove that any graph of n vertices and $t_{r-1}(n)+m$ edges, where $t_{r-1}(n)$ is the Turán number, contains $(1-o(1))m$ edge disjoint K_r's if $m=o(n^2)$. Furthermore, we determine the maximum m such that every graph of n vertices and $t_{r-1}(n)+m$ edges contains m edge disjoint K_r's if n is sufficiently large.

Introduction

In this paper, we study the number of edge disjoint K_r's in graphs of given order and size. We basically follow the notation of Bollobás [1]. E.g. $T_r(n)$ and $t_r(n)$ denotes the r-partite Turán graph of order n and its size, respectively. We denote by $d_G(v)$ and $N_G(v)$ the degree and the set of neighbours of v in G, respectively. A graph of order n and size m will sometimes be denoted by $G(n,m)$. The subgraph of G induced by $W \subseteq V(G)$ will be denoted by $G[W]$.

Let n and m be given integers. For a graph $G(n,m)$, let $edk_r(G)$ denote the maximum number of edge disjoint K_r's in G. Let $edk_r(n,m)$ denote the minimum of $edk_r(G)$ for the graphs G of order n and size m.

We conjecture that

$$edk_r(n,t_{r-1}(n)+m) \geq \left(\frac{2}{r}+o(1)\right)m.$$

The results of Erdős, Goodman and Pósa [4] ($r=3$) and Bollobás [2] ($r \geq 4$) imply that

$$edk_r(n,t_{r-1}(n)+m) \geq \frac{1}{\binom{r}{2}-1}m$$

Recently, Tuza and the present author [6] proved that

$$edk_3(n,t_2(n)+m) \geq \left(\frac{5}{9}+o(1)\right)m$$

and

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$$edk_r(n, t_{r-1}(n) + m) \geq \frac{1}{\binom{r}{2} - r/2} m \quad \text{if} \quad r \geq 4.$$

However, the conjecture is far from being best possible if m is not too large. Clearly,

$$edk_r(n, t_{r-1}(n) + m) \leq m.$$

(E.g. add m arbitrary edges to $T_{r-1}(n)$.) Erdős [3] proposed the problem of determining the maximum m such that $edk_3(n, t_2(n) + m) = m$. We proved in [5]

Theorem A.

$$edk_3(n, t_2(n) + m) = m \quad \text{if} \quad m \leq 2n - 10 \quad \text{for odd } n \text{ and}$$
$$\text{if} \quad m \leq \frac{3}{2}n - 5 \quad \text{for even } n$$

provided that n is sufficiently large.

Examples show that the upper bounds for m in Theorem A are sharp. (Unfortunately, I gave wrong upper bound $2n - 9$ and extreme graphs for odd n in [5]. The right extremal graphs are given in paper "Edge disjoint cliques in graphs" of the present author submitted to "Sets, Numbers and Graphs, Proceedings of the Colloquium dedicated to the 60th birthday of A. Hajnal and V. T. Sós.)

In this paper, we investigate $edk_r(n, t_{r-1}(n) + m)$ when $r \geq 4$ is arbitrary and $m = o(n^2)$. We will prove the following theorems

Theorem 1.

$$edk_r(n, t_{r-1}(n) + m) = m - O\left(\frac{m^2}{n^2}\right) = (1 - o(1))m \text{ if } m = o(n^2).$$

Remark. The weaker estimate $edk_r(n, t_{r-1}(n) + m) = (1 - o(1))m$ can be proved by means of a theorem of Lovász and Simonovits ([7], Thm. 2), as well.

Theorem 2.

$$edk_r(n, t_{r-1}(n) + m) = m \text{ if } m \leq 3\left[\frac{n+1}{r-1}\right] - 5 \text{ for } r \geq 4$$

provided that n is sufficiently large.

The following example shows that the upper bound for m in Theorem 2 is sharp.

Example. Consider two color classes V_i and V_j of $T_{r-1}(n)$ of $\left[\frac{n+1}{r-1}\right]$ vertices. Add $3\left[\frac{n+1}{r-1}\right] - 4$ edges to $T_{r-1}(n)$ so that two elements of V_i and one element of V_j should be joined to all vertices. If the $\left[\frac{n+1}{r-1}\right] - 1$ edges in V_j are covered by $\left[\frac{n+1}{r-1}\right] - 1$ edge disjoint K_r's the deleting these $\left(\left[\frac{n+1}{r-1}\right] - 1\right)\binom{r}{2}$ edges, every vertex in V_i has $\left[\frac{n+1}{r-1}\right] - 2$ neighbours in V_j except one which has $\left[\frac{n+1}{r-1}\right]$ neighbours. Since two elements of V_i have $\left[\frac{n+1}{r-1}\right] - 1$ neighbours in V_i, we cannot find $2\left[\frac{n+1}{r-1}\right] - 3$ edge disjoint K_r's covering the $2\left[\frac{n+1}{r-1}\right] - 3$ edges incident to these two vertices.

2. Proof of Theorem 1

The proof will be structurally similar to the proof of the case $r = 3$. We will prove that either we can find m edge disjoint K_r's or the graph is essentially isomorphic to a graph obtained from $T_{r-1}(n)$ by adding m edges. Then we will be able to complete the proof using the structure of the graph. ("Essentially" means that the size of the symmetric difference graph is small.) Throughout the proof, $r \geq 3$ is fixed and considered as a constant.

Let $G = (V, E)$ be an arbitrary graph of order n and size $t_{r-1}(n) + m$ where $m = o(n^2)$. Delete the edges of edge disjoint K_r's of maximum number. Let G^* denote the resulting K_r-free graph. If we found m edge disjoint K_r's then we are done. So we may assume that

$$|E(G^*)| \geq t_{r-1}(n) - \left[\binom{r}{2} - 1\right] m. \tag{1}$$

Choose a maximum size spanning $(r-1)$-partite subgraph G_o of G^* (with color classes $V_1, V_2, \ldots, V_{r-1}$) and let G_1 denote the spanning subgraph of the monochromatic edges i.e. $G_1 = (V,\ E(G^*) - E(G_o))$. Clearly,

$$|N_{G^*}(v) \cap V_i| \geq d_{G_1}(v) \tag{2}$$

for $v \in V$, $i = 1, \ldots, r-1$. (If it is not the case then deleting v from its color class and adding it to V_i, we obtain a spanning $(r-1)$-partite subgraph of greater size, a contradiction.

A fundamental theorem of Simonovits [8] (or see [1], p. 340.) says that a K_r-free graph of order n and size $t_{r-1}(n) - o(n^2)$ contains an $(r-1)$-partite graph of size $t_{r-1}(n) + o(n^2)$. This theorem implies that

$$|E(G_o)| = t_{r-1}(n) - o(n^2), \tag{3}$$

$$|E(G_1)| = o(n^2) \tag{4}$$

and (3) implies that

$$|V_i| = \frac{n}{r-1} + o(n) \quad \text{for} \quad i = 1, \ldots, r-1. \tag{5}$$

Proposition 1. $d_{G_1}(v) = o(n)$ for $v \in V$.

Proof. Suppose that there is a vertex $v \in V$ such that $d_{G_1}(v) = p \geq \varepsilon n$ for some constant $\varepsilon > 0$. Then there are sets $V_i' \subseteq V_i (i = 1, \ldots, r-1)$ such that $|V_i'| = p$, $V_i' \subseteq N_{G^*}(v)$ by (2). Let us consider the $(r-1)$-partite subgraph $G_o[V_1' \cup \cdots \cup V_{r-1}']$. It does not contain any K_{r-1} since then the addition of v to this K_{r-1} results a K_r in G^*, a contradiction. Thus every K_{r-1} in the complete $(r-1)$-partite graph of color classes $V_1', \ldots, V_{r-1}'$ is represented by some edge missing from $G_o[V_1' \cup \cdots \cup V_{r-1}']$. However one missing edge represents $p^{r-3} K_{r-1}$'s from among the $p^{r-1} K_{r-1}$'s. Thus, at least p^2 edges are missing and so

$$|E(G_o)| \leq \sum_{1 \leq i < j \leq r-1} |V_i||V_j| - p^2 \leq t_{r-1}(n) - \varepsilon^2 n^2,$$

a contradiction to (3). ∎

Before the following basic lemma, which will be used several times, we need a definition.

Definition. For a coloring $\{V_1, \dots, V_{r-1}\}$, a clique K_{r-1} (K_r) is said to be totally multicolored if $V(K_{r-1}) \cap V_i \neq \emptyset$ ($V(K_r) \cap V_i \neq \emptyset$) for $i = 1, \dots, r-1$.

Lemma 2. Let H be the union of an $(r-1)$-partite graph H_o of order n with color classes $W_1, \dots, W_{r-1}$ and p independent monochromatic edges $x_1y_1, \dots, x_py_p$ in some color class W_i. Suppose that

$$|E(H_o)| \geq t_{r-1}(n) - c_1 m + p \cdot o(n) \tag{6}$$

for some constant c_1 and $m = o(n^2)$. Then for any $\varepsilon > 0$, there is a constant $c_o(\varepsilon)$ such that H contains $(1-\varepsilon)p$ edge disjoint totally multicolored K_r's if $p > c_o \frac{m}{n}$ and n is sufficiently large.

Furthermore, if for $j \neq i$, we fix some vertex sets $W'_j \subseteq W_j$ such that $|W'_j| > p$, $|W'_j| \geq \frac{n}{2(r-1)}$ then these K_r's can be chosen so that they should be contained in $H\left[\bigcup_{j\neq i} W'_j \cup \{x_1, y_1, \dots, x_p, y_p\}\right]$.

Proof. First we prove a weaker statement, namely, that H contains one desired K_r. Suppose not. Inequality (6) implies (3) for H_o and it implies (5) for $W_1, \dots, W_{r-1}$ and so $p \leq \frac{n}{2(r-1)} + o(n)$. Consider some vertex sets $W''_j \subseteq W'_j$ (or simply $W''_j \subseteq W_j$ if W'_j is not defined) such that $|W''_j| = q > p$, $q \geq \frac{n}{2(r-1)}$ for $j \neq i$ and the vertex set $W''_i = \{x_1, y_1, \dots, x_p, y_p\}$. Add the edges $x_1y_1, \dots, x_py_p$ to the complete $(r-1)$-partite graph with color classes $W''_1, \dots, W^n_{r-1}$ and let H^* denote the resulting graph. The graph H^* contains pq^{r-1} K_r's and every multicolored edge of it represents either q^{r-3} or pq^{r-4} ($\leq q^{r-3}$) K_r's. Since H does not contain any K_r thus H must miss at least $pq^{r-2}/q^{r-3} = pq$ multicolored edges of H^*. Thus

$$|E(H_o)| \leq t_{r-1}(n) - pq \leq t_{r-1}(n) - \frac{pn}{2(r-1)} \tag{7}$$

Combining (6) and (7), we have

$$t_{r-1}(n) - c_1 m + p \cdot o(n) \leq t_{r-1}(n) - \frac{pn}{2(r-1)}.$$

It follows that

$$p \leq c_1(r-1)(2 + o(1))\frac{m}{n}$$

which is a contradiction if $c_o > 2c_1(r-1)$.

We proved that H contains a K_r. Delete its edges from H and apply the proved weaker statement for the remaining graph, and then for the next remaining graph, etc. How long can we do it? Suppose that we applied it $k < (1-\varepsilon)p$ times and let H_k denote the remaining $(r-1)$-partite graph to the color class W_i of which $p-k$ independent edges are added now. The inequality (6) implies that

$$|E(H_k)| \geq t_{r-1}(n) - c_1 m - k\left[\binom{r}{2} - 1\right] + p \cdot o(n). \tag{6'}$$

Also, we have

$$(7') \qquad |E(H_k)| \leq t_{r-1}(n) - (p-k)\frac{n}{2(r-1)}.$$

Combining (6') and (7'), we have

$$c_1 m + k\left[\binom{r}{2} - 1\right] + p \cdot o(n) \geq (p-k)\frac{n}{2(r-1)}.$$

If follows that

$$k\left[\frac{n}{2(r-1)} + \binom{r}{2} - 1\right] \geq p\left[\frac{n}{2(r-1)} + o(n)\right] - c_1 m,$$

and using $k < (1-\varepsilon)p$, we have

$$(1-\varepsilon)p\left[\frac{n}{2(r-1)} + \binom{r}{2} - 1\right] > p\left[\frac{n}{2(r-1)} + o(n)\right] - c_1 m.$$

This implies that

$$p\left[\frac{\varepsilon}{2(r-1)} + o(1)\right] < c_1 \frac{m}{n},$$

a contradiction if $c_o > \frac{2c_1(r-1)}{\varepsilon}$ and n is sufficiently large. ∎

Let p denote the maximum number of independent edges in G_1 in the same color class. Then at most $2p(r-1)$ vertices represent all edges of G_1, and so

$$(8) \qquad |E(G_1)| \leq p \cdot o(n)$$

by Proposition 1. Considering that G^* is K_r-free, the inequalities (1) and (8) imply that (6) holds for G_o with $c_1 = \binom{r}{2} - 1$. Using Lemma 2, we have the following

Proposition 3. The graph G_1 contains at most $O(1)\frac{m}{n}$ independent edges and so there are $O(1)\frac{m}{n}$ vertices representing all edges of G_1.

Now using Lemma 2, we estimate the number of vertex disjoint not totally multicolored K_r's deleted from G.

Proposition 4. The number of vertex disjoint not totally multicolored K_r's deleted from G is $O(1)\frac{m}{n}$ and so there are $O(1)\frac{m}{n}$ vertices representing all not totally multicolored K_r's deleted from G.

Proof. Suppose that we deleted $q \geq c\frac{m}{n}$ vertex disjoint not totally multicolored K_r's where the constant c will be determined later. Consider a representing monochromatic edge e_i $(i = 1, \ldots, q)$ of each of these K_r's. Say, V_1 contains at least $\frac{q}{r-1}$ edges from among these independent edges. Applying Lemma 2 for G_o and these edges, we obtain that the union of G_o and these edges contains $(1-\varepsilon)\frac{q}{r-1}$ totally multicolored K_r's. For the edges e_i that are contained in these $(1-\varepsilon)\frac{q}{r-1}$ totally multicolored K_r's, add back to G^* the edge set of the deleted K_r containing e_i and delete the edge set of the totally multicolored K_r containing e_i. Now, consider another representing monochromatic edge f_i of each of the K_r's added back to G^*. (Each of these K_r's contains at least one more monochromatic edge since these K_r's are not totally multicolored.) These are $(1-\varepsilon)\frac{q}{r-1} \geq \frac{(1-\varepsilon)c}{r-1} \cdot \frac{m}{n}$ independent edges. If this new G^* is K_r-free then it contradicts Proposition 3 if c is large enough and if this new G^* contains some K_r then it contradicts the maximum choice of the deleted edge disjoint K_r's. ∎

Propositions 3 and 4 imply that there is a vertex set $\{v_1, \ldots, v_q\}$ of size $O(1)\frac{m}{n}$ representing all edges of G_1 and all not totally multicolored K_r's deleted from G.

For a vertex $v \in V$ and for a coloring $\{W_1, \ldots, W_{r-1}\}$ of V let

$$d_{mon}(v; W_1, \ldots, W_{r-1}) = d_{G_1}(v) + \# \begin{array}{l}\text{monochromatic edges incident}\\ \text{to } v \text{ that are contained in some}\\ \text{not totally multicolored } K_r\\ \text{deleted from } G\end{array}$$

Now we recolor the vertices $v_1, \ldots, v_q$ so that $\sum_{x \in V} d_{mon}(x; V_1', \ldots, V_{r-1}')$ should be minimum for the resulting coloring $\{V_1', \ldots, V_{r-1}'\}$. Let G_1' denote the graph of monochromatic edges in coloring $\{V_1', \ldots, V_{r-1}'\}$ and let $G_0' = (V, E(G^*) - E(G_1'))$. Let

$$d = \max_{x \in V} d_{mon}(x; V_1', \ldots, V_{r-1}').$$

Notice that the vertices $v_1, \ldots, v_q$ still represent all monochromatic edges (the edges of G_1') and all not totally multicolored K_r's deleted from G. Thus, G contains $O(1)qd$ monochromatic edges contained either in G_1' or in deleted not totally multicolored K_r's. In addition, the $m'(< m)$ deleted totally multicolored K_r's contain m' monochromatic edges. So, the graph G has at most $O(1)q \cdot d + m' < O(1)\frac{m}{n}d + m$ monochromatic edges and so at least $t_{r-1}(n) + O(1)\frac{m}{n}d$ multicolored edges.

Now, we will prove that $d = O(1)\frac{m}{n}$. We prove this statement by contradiction. Suppose that $d > c\frac{m}{n}$ where the constant c will be determined later. Let $x \in V$ be a vertex such that $d_{mon}(x; V_1', \ldots, V_{r-1}') = d$. Notice that $d_{mon}(v; V_1', \ldots, V_{r-1}') \leq q(r-1)$ for $v \in V - \{v_1, \ldots, v_q\}$ since $\{v_1, \ldots, v_q\}$ represents all edges of G_1' and all not totally multicolored K_r's deleted from G. Thus we may assume that $x \in \{v_1, \ldots, v_q\}$. Without loss of generality, we may assume that $x \in V_1'$. First we prove

Proposition 5.

$$d_{mon}(x; V_1' - \{x\}, \ldots, V_{i-1}', V_i' \cup \{x\}, V_{i+1}', \ldots, V_{r-1}') \geq \frac{d}{2}$$

for $i = 2, \ldots, r-1$.

Proof. For shortness, the coloring $\{V_1' - \{x\}, \ldots, V_{i-1}', V_i' \cup \{x\}, V_{i+1}', \ldots, V_{r-1}'\}$ will be denoted by $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ including $\{V_1'(x,1), \ldots, V_{r-1}'(x,1)\} = \{V_1', \ldots, V_{r-1}'\}$.

We know that

$$\sum_{v \in V} d_{mon}(v;\ V_1', \ldots, V_{r-1}') \leq \sum_{v \in V} d_{mon}(v;\ V_1'(x,i), \ldots, V_{r-1}'(x,i))$$

by the choice of coloring $\{V_1', \ldots, V_{r-1}'\}$. Table 1 shows how the terms of $d_{mon}(x; V_1', \ldots, V_{r-1}')$ and $\sum_{v \in V} d_{mon}(v;\ V_1', \ldots, V_{r-1})$ will change when we turn to the coloring $V_1'(x,i), \ldots, V_{r-1}'(x,i)$ $(i \geq 2)$.

Term	Change in $dmon(x, V_1', \ldots, X_{r-1}')$	Change in $\sum_{v \in V} dmon(v; V_1', \ldots, X_{r-1}')$
$d_{G_1}(x)$	$+\lvert N_{G^*}(x) \cap V_i' \rvert - d_{G_1'}(x)$	$+2\lvert N_{G^*}(x) \cap V_i' \rvert - 2d_{G_i'}(x)$
K_r such that $x \in V(K_r)$, K_r is not multicolored in $\{V_1', \ldots, V_{r-1}'\}$ K_r is not totally multicolored in $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$	$+\lvert V(K_r) \cap V_i' \rvert - \lvert V(K_r) \cap V_1' \rvert + 1$	$+2\lvert V(K_r) \cap V_i' \rvert - 2\lvert V(K_r) \cap V_1' \rvert + 2$
K_r such that $x \in V(K_r)$, K_r is not totally multicolored in $\{V_1', \ldots, V_{r-1}'\}$ K_r is totally multicolored in $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ $\lvert V(K_r) \cap V_1' \rvert = 2$	-1	-4
K_r such that $x \in V(K_r)$, K_r is not totally multicolored in $\{V_1', \ldots, V_{r-1}'\}$ K_r is totally multicolored in $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ $\lvert V(K_r) \cap V_1' \rvert = 3$	-2	-6
K_r such that $x \in V(K_r)$, K_r is totally multicolored in $\{V_1', \ldots, V_{r-1}'\}$ K_r is not totally multicolored in $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ $\lvert V(K_r) \cap V_1' \rvert = 1$	$+1$	$+4$
K_r such that $x \in V(K_r)$, K_r is totally multicolored in $\{V_1', \ldots, V_{r-1}'\}$ K_r is not totally multicolored in $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ $\lvert V(K_r) \cap V_1' \rvert = 2$	$+2$	$+6$
K_r such that $x \in V(K_r)$, and K_r is totally multicolored in $\{V_1', \ldots, V_{r-1}'\}$ and $\{V_1'(x,i), \ldots, V_{r-1}'(x,i)\}$ or $x \notin V(K_r)$	0	0

Table 1

Table 1 shows that if a term of $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$ decreases $\triangle$ then the sum decreases at least $2\triangle$ and if a term of $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$ increases $\triangle$ then the sum increases at most $4\triangle$. Considering that the sum did not decrease by the change of the color of x, it implies that the increasing terms of $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$ increased at least half as much as the decreasing terms of $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$ decreased. On the other hand, the sum of decreases of the decreasing terms of $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$ is at most $d_{mon}(x;\ V'_1, \dots, V'_{r-1})$, since all terms are nonnegative at any time. Thus

$$d_{mon}(x;\ V'_1(x,i), \dots, V'_{r-1}(x,i)) \geq \frac{1}{2} d_{mon}(x;\ V'_1, \dots, V'_{r-1}). \qquad \blacksquare$$

Proposition 6. We may assume that

$$|N_{G^*}(x) \cap V'_i| > \frac{d}{4r^2} \quad \text{for} \quad i = 1, \dots, r-1.$$

In particular, it yields that $d_{G'_1}(x) > \frac{d}{4r^2}$.

Proof. Suppose that

$$|N_{G^*}(x) \cap V'_i| \leq \frac{d}{2r} \tag{9}$$

for some index $1 \leq i \leq r-1$. We have

$$d_{mon}(x;\ V'_1(x,i), \dots, V'_{r-1}(x,i)) \geq \frac{d}{2} > \frac{c}{2}\frac{m}{n} \tag{10}$$

by the choice of x if $i = 1$ or by Proposition 5 if $2 \leq i \leq r-1$. Inequalities (9) and (10) imply that in coloring $\{V'_1(x,i), \dots, V'_{r-1}(x,i)\}$, there are $\frac{d}{2} - \frac{d}{2r} = \frac{d(r-1)}{2r}$ monochromatic edges incident to x that are contained in not totally multicolored K_r's deleted from G. Thus, at least $\frac{d}{2r}$ not totally multicolored deleted K_r's contain some monochromatic edge incident to x in coloring $\{V'_1(x,i), \dots, V'_{r-1}(x,i)\}$. Let $H_1, \dots, H_s$ ($s = \lceil \frac{d}{2r} \rceil$) be such deleted K_r's and let $x_jy_j \in E(H_j)$ be a monochromatic edge not incident to x for $j = 1, \dots, s$. Notice that the edges x_jy_j are independent, since the deleted K_r's are edge disjoint and $x \in V(H_j)$ for $j = 1, \dots, s$. Now there is a color class V'_{io} containing at least $\lceil s/(r-1) \rceil > \frac{d}{2r(r-1)}$ edges x_jy_j, say x_jy_j for $j = 1, \dots, \lceil s/(r-1) \rceil$. Here, we can apply Lemma 2 for the union of G'_o and the edges x_jy_j ($1 \leq j \leq \lceil s/(r-1) \rceil$) since G_o satisfies (6). To produce G'_o we recolored $O(1)\frac{m}{n}$ vertices and so $|E(G_o)|$ changes $O(1)m$ i.e. G'_o also satisfies (6) with some other c_1. So by Lemma 2, the union of G'_o and these $\lceil s/(r-1) \rceil$ independent edges contains $\frac{1}{2}\lceil s/(r-1) \rceil > \frac{d}{4r(r-1)}$ edge disjoint K_r's not containing x. For the edges x_jy_j contained in these $\frac{1}{y}\lceil s/(r-1) \rceil$ K_r's, add back to G^* the originally deleted K_r containing x_jy_j and delete this new K_r which does not contain x. For the resulting graph G^*, we have the inequality

$$|N_{G^*}(x) \cap V'_i| > \frac{d}{4r^2}.$$

Notice that during this procedure, $E(G^*)$ and $E(G'_o)$ changes $O(1)m$ and so Lemma 2 can be successively applied for all sets V'_i satisfying (9). ∎

Now we are ready to finish the proof of the estimate $d = O(1)\frac{m}{n}$. By Proposition 6, we may assume that $N_{G^*}(x)$ contains some vertex sets $W_1 \subseteq V'_1, \ldots, W_{r-1} \subseteq V'_{r-1}$ such that $|W_1| = \cdots = |W_{r-1}| = t = \lceil \frac{d}{4r^2} \rceil$. Let $W = W_1 \cup \cdots \cup W_{r-1}$. We can see that $t = o(n)$ and that $G'_o[W]$ misses at least $t^2 \geq \frac{d^2}{16r^4}$ multicolored edges (see the proof of Proposition 1). Let $H_1, \ldots, H_k$ denote the deleted K_r's containing some W_iW_j-edge. We will prove that we may assume that $k = O(1)t\frac{m}{n} = O(1)d\frac{m}{n}$. Suppose that $k > c_2t\frac{m}{n}$ where the constant c_2 will be determined later. Let us choose some representing vertex $u_i \in V(H_i)$ for $i = 1, \ldots, k$ such that $u_i \in W$ and u_i does not represent all monochromatic edges of H_i. (It can be done since $V(H_i)$ meets at least two W_i's.) Further, let x_iy_i be a monochromatic edge of H_i not incident to u_i for $i = 1, \ldots, k$. Then there is a vertex $u \in W$ which represents $s \geq \frac{k}{|W|} > \frac{c_2t\frac{m}{n}}{(r-1)t} = \frac{c_2}{r-1}\frac{m}{n}$ deleted H_i's, say $H_1, \ldots, H_s$. The monochromatic edges $x_1y_1, \ldots, x_sy_s$ are independent and there is a color class V'_{i_o} containing at least $\lceil \frac{s}{r-1} \rceil > \frac{k}{(r-1)^2}t$ from among them, say, $x_1y_1, \ldots, x_{\lceil \frac{s}{r-1} \rceil}y_{\lceil \frac{s}{r-1} \rceil}$. If c_2 is large enough then here we can apply Lemma 2 for the union of G'_0 (or later in the procedure some new G'_o) and these $\lceil \frac{s}{r-1} \rceil$ independent edges since G'_o satisfies (6) and in these steps only the deleted K_r's change. Thus $|E(G'_o)|$ changes $O(1)m$ and each new G'_o also satisfies (6) with some other constant c_1. By Lemma 2, the union of G'_o and these $\lceil \frac{s}{r-1} \rceil$ independent edges contains $\frac{1}{2}\lceil \frac{s}{r-1} \rceil \geq \frac{k}{2(r-1)^2t}$ edge disjoint K_r's such that the vertex set of any of these K_r's does not meet $W \cup \{x\}$ apart from the two endvertices x_i and y_i when x_iy_i is an edge of it. For the edges x_iy_i contained in these $\frac{1}{2}\lceil \frac{s}{r-1} \rceil$ K_r's, add H_i back to G^* and delete this new K_r which contains x_iy_i but does not contain any other edge of $G[W \cup \{x\}]$. We can continue this procedure as long as we have $k = O(1)d\frac{m}{n}$. So we may assume that the deleted K_r's contain $O(1)d\frac{m}{n}$ edges of the complete $(r-1)$-partite graph with color classes $W_1, \ldots, W_{r-1}$.

Recall that G misses $O(1)d\frac{m}{n}$ multicolored edges and so $G'_o[W]$ also misses $O(1)d\frac{m}{n}$ multicolored edges which contradicts the fact that $G'_o[W]$ misses at least $t^2 > \frac{d^2}{16r^4} > \frac{cd}{16r^4}\frac{m}{n}$ multicolored edges if c is sufficiently large.

We proved that $d = O(1)\frac{m}{n}$. It follows by the definition of d and Proposition 4 that the number of deleted not totally multicolored K_r's is $O(1)d\frac{m}{n} = O(1)\frac{m^2}{n^2}$ and so is the number of monochromatic edges contained in the deleted not totally multicolored K_r's. Similarly, it follows by the definition of d and Proposition 3 that $|E(G'_1)| = O(1)d\frac{m}{n} = O(1)\frac{m^2}{n^2}$.

Since G has $t_{r-1}(n) + m$ edges, it must contain at least m monochromatic edges. Using the two facts above, it implies that $m + O(1)\frac{m^2}{n^2}$ monochromatic edges are contained in totally multicolored deleted K_r's each of which contains exactly one monochromatic edge. I.e., we deleted and thus G contained $m + O(1)\frac{m^2}{n^2}$ edge disjoint K_r's what we wanted to prove. ∎

Notice that in the proof of Theorem 1, we also proved the following statement which will be used in the proof of Theorem 2.

Theorem 3. *Let G be a graph of order n and size $t_{r-1}(n)+m$, $m=o(n^2)$. Then either G contains m edge disjoint K_r's or there is an $(r-1)$-coloring $\{V_1,\dots,V_{r-1}\}$ of $V(G)$ such that the number of not monochromatic edges is $t_{r-1}(n)+O(1)\frac{m^2}{n^2}$ and so $|V_i|=\frac{n}{r-1}+O(1)\frac{m}{n}$ for $i=1,\dots,r-1$.*

3. Proof of Theorem 2

The following lemma will be fundamental in the proof.

Lemma 7. Let H be an $(r-1)$-partite graph of order s with color classes $W_1,\dots,W_{r-1}$ such that $|W_1|\le|W_2|=|W_3|=\dots=|W_{r-1}|$. Suppose that

$$(11)\qquad |N_H(w)\cap W_j|\ge\left(1-\frac{1}{3r}\right)|W_j|\quad\text{for}\quad j\ge 2, w\in W_i,\ i\ne j$$

and

$$(12)\qquad |N_H(w)\cap W_1|\ge\left(1-\frac{1}{3r}\right)|W_1|\quad\text{for}\quad w\notin W_1\quad\text{if}\quad |W_1|\ge\frac{1}{2}|W_2|.$$

Then there exist $|W_1|$ vertex disjoint K_{r-1}'s in H (which obviously cover W_1) if s is sufficiently large.

Proof. We prove the lemma by induction on r. The statement obviously holds for $r=2$. Suppose that the statement holds for $r\le r_o$ ($r_o\ge 2$) and let H be an r_o-partite graph with color classes $W_1,\dots,W_{r_o}$ satisfying the conditions of the lemma. Then the (r_o-1)-partite graph $H_o=H[W_1\cup\dots\cup W_{r_o-1}]$ obviously satisfies the conditions of the lemma with $r=r_o$. Thus there $|W_1|$ vertex disjoint K_{r_o-1}'s in H_o by the inductional hypothesis. Now, we define a bipartite graph H^* with color classes W_1 and W_{r_o}. The vertices $x\in W_1$ and $y\in W_{r_o}$ should be joined by an edge if and only if the addition of y to the K_{r_o-1} containing x results in a K_{r_o} in H, i.e. y is adjacent to all vertices of the K_{r_o-1} containing x.

To prove Lemma 7, it is sufficient to prove that H^* contains a matching covering W_1. Thus according to P. Hall's well-known matching theorem, it is sufficient to verify Hall's conditions

$$(13)\qquad \left|\bigcup_{x\in X}N_{H^*}(x)\right|\ge|X|\quad\text{for}\quad X\subseteq W_1.$$

The lower bound (11) obviously implies that

$$(14)\qquad |N_{H^*}(w)|\ge\frac{2r_o+1}{3r_o}|W_{r_o}|\quad\text{for}\quad w\in W_1,$$

which implies that H^* satisfies Hall's condition if $|W_1|\le\frac{1}{2}|W_{r_o}|$ and s is sufficiently large.

Suppose that $|W_1|\ge\frac{1}{2}|W_{r_o}|$. Then (11) and (12) imply that

$$(15)\qquad |N_{H^*}(y)|\ge\left(1-\frac{1}{3r_o}\right)|W_1|-\frac{r_o-2}{3r_o}|W_{r_o}|>\frac{1}{3}|W_1|\quad\text{for}\quad y\in W_{r_o}.$$

On the other side, (14) implies that if $|X| \le \frac{2}{3}|W_{r_o}|$ ($X \subseteq W_1$) then X satisfies Hall's condition. Suppose that $|X| > \frac{2}{3}|W_{r_o}| \ge \frac{2}{3}|W_1|$. Then (15) implies that $N_{H^*}(y) \cap X \ne \emptyset$ for $y \in W_{r_o}$ and $\bigcup_{x \in X} N_{H^*}(x) = W_{r_o}$, Hall's condition is satisfied for X in this case, as well. ∎

Now we are ready to prove Theorem 2 for $r \ge 4$. Let G be a graph of order n and size $t_{r-1}(n) + m$, $m \le 3\left[\frac{n+1}{r-1}\right] - 5$. As Theorem 3 states, we may assume that $V(G)$ has an $(r-1)$-coloring $\{V_1, \dots, V_{r-1}\}$ such that the number of multicolored edges is $t_{r-1}(n) + O(1)$ and so $|V_i| = \frac{n}{r-1} + O(1)$ for $i = 1, \dots, r-1$. Let G_1 denote the spanning subgraph of the monochromathic edges in G and let m' denote its size. We have $m' = \frac{3n}{r-1} + O(1)$. We label the vertices of G so that

$$d_i(v_i) = d_{G_1[V(G)-\{v_1,\dots,v_{i-1}\}]}(v_i) = \\ = \max_{x \in V(G)-\{v_1,\dots,v_{i-1}\}} d_{G_1[V(G)-\{v_1,\dots,v_{i-1}\}]}(x).$$

Clearly, $\sum_{i=1}^{n} d_i(v_i) = m'$ and $d_1(v_1) \ge d_2(v_2) \ge \cdots \ge d_n(v_n) = O$.

A bit long, but straightforward case by case analysis shows that if G contains all multicolored edges (misses one and at least two multicolored edges, resp.) then G contains $\sum_{j=1}^{3} d_j(v_j)$ ($\sum_{j=1}^{3} d_j(v_j) - 1$ and $\sum_{j=1}^{3} d_j(v_j) - 2$, resp.) edge disjoint K_r's whose monochromatic edges are incident to v_1, v_2 or v_3.

Let $i \ge 4$. Delete from G the edge disjoint K_r's found so far and delete the edges of G_1 incident to v_1, v_2 or v_3 even if they are not covered by K_r's found so far. Let G^i denote the resulted graph and let G_1^i denote the spanning subgraph of the monochromatic edges in G^i. (In this notation, we have $d_i(v_i) = d_{G_1^i}(v_i)$.) In G^i, we will find $d_i(v_i)$ edge disjoint K_r's whose monochromatic edges are incident to v_i. Deleting these $d_i(v_i)$ K_r's from G^i, we obtain G^{i+1}.

Consider the next index i. (We start with $i = 4$.) Let W_1 be $N_{G_1^i}(v_i)$ and let W_j' be $N_{G^i}(v_i) \cap V_j$ for the indices $2 \le j \le r-1$ such that $v_i \notin V_j$. If we find $|W_1| = d_i(v_i)$ edge disjoint totally multicolored K_{r-1}'s in $G^i[W_1 \cup \bigcup_{\ell=2}^{r-1} W_\ell']$ then adding v_i to them, we obtain the desired K_r's. Notice that

$$|W_1| \le \frac{\sum_{j=1}^{i} d_j(v_j)}{i} \le \frac{m'}{4} = \frac{3n}{4(r-1)} + O(1).$$

To define W_ℓ for $2 \le \ell \le r-1$, in the first step, delete from W_ℓ' the vertices v such that $|N_{G^i}(v) \cap V_j| \le \frac{n}{r-1} - \frac{n}{4r^2}$ for some $j \in \{1, \dots, r-1\}$ such that $v \notin V_j$. Now, if their size are different then delete the appropriate number of vertices from the too large sets. Notice that we delete $O(1)$ vertices from W_ℓ' to obtain W_ℓ ($2 \le \ell \le r-1$) since G^i misses at most $O(1)n$ multicolored edges and if we have to delete v from

W'_ℓ in the first step then at least $\frac{n}{4r^2}+O(1)$ missing multicolored edges are incident to v.

Recall that $|N_G(v)\cap V_j|=\frac{n}{r-1}+O(1)$ if $v\notin V_j$. Now, we state

Proposition 8.

$$(16)\qquad |N_{G^i}(v)\cap V_j|\le \frac{n}{r-1}-\frac{n}{4r^2}+O(1)\quad\text{if}\quad v\notin V_j,\quad v\notin\{v_1,\dots,v_{i-1}\}.$$

Proof. Suppose that there is an index $i_o\le i$ such that

$$|N_{G^{i_o-1}}(v)\cap V_j|\ge \frac{n}{r-1}-\frac{n}{4r^2}$$

and

$$|N_{G^{i_o}}(v)\cap V_j|< \frac{n}{r-1}-\frac{n}{4r^2}$$

At one step, $|N_{G^k}(v)\cap V_j|\quad (k\le i_o-1)$ decreases at most two thus we have

$$i_o\ge\frac{n}{8r^2}+O(1)$$

and so

$$d_{i_o}(v_{i_o})\le\frac{1}{i_o}\sum_{j=1}^{i_o}d_j(v_j)\le\frac{m'}{i_o}=O(1).$$

Therefore, we have

$$d_{G_1^{i_o}}(v)=O(1)$$

by the definition of v_{i_o}. Suppose we had $v\in N_{G^k}(v_k)\cap V_j$ for $k\ge i_o$. If $v_k\in V_j$ then we deleted v in the definition of $W_\ell\subseteq V_j$ and $v\in N_{G_1^k}(v_k)$ could occur at most $d_{G_1^{i_o}}(v)=O(1)$ times. Thus, we have

$$|N_{G^i}(v)\cap V_j|\ge\frac{n}{r-1}-\frac{n}{4r^2}+O(1),$$

what we wanted to prove. ∎

Now (16) implies that when define W_ℓ for $\ell\ge 2$ then we delete at most $12r^2=O(1)$ vertices. Thus, we will have

$$(17)\qquad |W_\ell|\ge\frac{n}{r-1}-\frac{n}{4r^2}+O(1)\quad\text{for}\quad \ell=2,\dots,r-1$$

and we have $|W_1|\le|W_2|$ if n is sufficiently large. Estimate (17) implies that if $|W_1|\ge\frac{1}{2}|W_2|$ then $d_i(v_i)\ge cn$ for some constant c and so $i=O(1)$. Hence,

$$|N_{G^i}(v)\cap W_1|\ge|W_1|+O(1)\quad\text{for}\quad v\in W_\ell\ell\ge 2$$

which implies (12). Moreover if $v\in W_k$ then either Proposition 8 ($k=1$) or the definition of W_k ($k\ge 2$) implies that for $\ell=2,\dots,r-1$, $\ell\ne k$

$$|N_{G^i}(v)\cap W_\ell|\ge|W_\ell|-\frac{n}{4r^2}+O(1)\ge$$

$$\ge\left(1-\frac{1}{3r}\right)|W_\ell|+\frac{|W_\ell|}{3r}-\frac{n}{4r^2}+O(1)\ge\left(1-\frac{1}{3r}\right)|W_\ell|$$

if n is sufficiently large. So (11) holds and applying Lemma 7, we can find $d_i(v_i)$ desired K_r's, which completes the proof of Theorem 2. ∎

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